Math 10B with Professor Stankova
Worksheet, Discussion #28; Thursday, 5/3/2018
GSI name: Roy Zhao

1 Permutations/Combinations/12-Fold Way

1.1 Counting

1. How many ways are there to rearrange the letters of $ZVEZDA$?

> **Solution:** $\frac{6!}{2!} = 360$ where we divide by 2! because there are 2 Z's.

2. What is the coefficient of $x^2y^3z^5$ in $(x/2+3y-2z)^{10}$? What about the coefficient of $x^3y^3z^3$?

> **Solution:** $\binom{10}{2,3,5}(\frac{1}{2})^2(3)^3(-2)^5 = \binom{10}{2}\binom{8}{3}\binom{5}{5}(27\cdot -8) = \frac{10!}{2!3!5!}\cdot 27\cdot(-8)$.

3. How many ways can I create a license plate that has 3 letters followed by 3 numbers if I want exactly 1 I and at least 1 1.

> **Solution:** First we count the number of ways we can choose 3 letters with exactly 1 I. The I can be in the first, second, or third spot. Each of the other options can be anything other than I and hence there are $1\cdot 25\cdot 25 + 25\cdot 1\cdot 25 + 25\cdot 25\cdot 1 = 3\cdot 25^2$ different ways to choose the 3 letters.
>
> Now we count the number of ways to pick the numbers. We see at least so we think complementary counting. There are 9^3 ways to choose 0 1's so there are 10^3-9^3 ways to choose at least 1 1. Therefore, there are $(3\cdot 25^2)(10^3-9^3)$ different license plates.

4. How many 5 card hands out of a standard 52 card deck have 4 different suits?

> **Solution:** If we want all 4 suits, 3 of the suits will have 1 card and one will have two cards in that suit. There are $\binom{4}{1} = 4$ ways to choose the suit that has 2 cards. Then for that suit, there are $\binom{13}{2}$ ways to choose the two cards in that suit. For the three

remaining suit, we just need to pick one card in that suit and there are $\binom{13}{1} = 13$ ways to do this. So, the number of hands is

$$\binom{4}{1}\binom{13}{2}\binom{13}{1}^3.$$

5. I am baking cookies for Alice Bob and Carol. Each want at least 1 cookie but Bob wants at least 3 cookies. Alice is on a diet and wants at most 3 cookies. How many ways can I divide the 10 cookies I made amongst them?

Solution: Let x_1, x_2, x_3 be the number of cookies they get. Then our conditions are that $1 \leq x_1 \leq 3, 3 \leq x_2, 1 \leq x_3$. We can think of this as giving Alice and Carol 1 cookie and Bob 3 cookies so we have 5 cookies left to distribute and now Alice can only get two more cookies. So we have the problem $y_1 + y_2 + y_3 = 5$ with $0 \leq y_1 \leq 2$. We can do this by letting $y_1 = 0, 1, 2$ and calculating the number of ways for each and summing them together. Another way to do this is counting the total number of ways without the restriction $y_1 \leq 2$ to get $\binom{5+3-1}{5} = \binom{7}{5}$ ways, and then subtracting the number of bad ways, which is when $y_1 \geq 3$. If $y_1 \geq 3$, we can subtract 3 to get $y_1' + y_2 + y_3 = 2$ and there are $\binom{2+3-1}{2} = \binom{4}{2}$ ways for this to happen. This gives a total of $\binom{7}{5} - \binom{4}{2} = 21 - 6 = 15$ total ways.

6. In the previous cookie problem, let X be the most number of cookies any one of Alice, Bob, or Carol gets (if they got $2, 3, 5$ cookies, then $X = 5$), what can we say about the minimum X can be?

Solution: By Pigeonhole Principle, by giving out 10 cookies to 3 people, there exists someone who gets at least $\lceil 10/3 \rceil = 4$ cookies. So, X must be at least 4.

1.2 Probability/Expected Value

7. Let X be a random variable on a probability space Ω with a probability function P and let f be the PMF for X. Draw a picture of how all these variables interact and explain any special arrows that you have in your diagram.

Solution: P is a dashed line from Ω to $[0, 1]$ because it takes in subsets of Ω. X is a solid arrow from Ω to $\mathbb{R}$ because it is a function that takes outcomes to a value. f is a solid arrow from $\mathbb{R}$ to $[0, 1]$.

8. When I roll a fair 6 sided die 10 times, what is the expected number of distinct numbers that appear? (For instance, if I roll $1, 1, 3, 3, 2$, there are 3 distinct numbers that appear)

Solution: Let $X_1, X_2, \dots, X_6$ be random variables such that $X_i = 1$ if I roll an i in the 10 times and 0 otherwise. Then the number of distinct numbers that appear is $E[X_1 + X_2 + \cdots + X_6] = E[X_1] + E[X_2] + \cdots + E[X_6]$. Then $E[X_1]$ is just the probability that I roll at least 1 1 in 10 roll. We calculate this via complementary counting because it says at least. This probability is $1 - P(A)$ where A is the event that I roll 0 1s. The probability of this is $P(A) = (\frac{5}{6})^{10}$. So the expected number of distinct numbers is $6(1 - (5/6)^{10})$.

9. Eve has 5 cards in her hand and I know that one of them is the ace of spades. What is the probability that she has a pair of aces (exactly 2 aces)?

Solution: Let A be the event that she has a pair of aces and let B be the event that she has the ace of spades. Then we want to calculate $P(A|B)$. By definition, this is $\frac{P(A\cap B)}{P(B)}$. The probability $P(B) = \frac{\binom{51}{4}}{\binom{52}{5}}$ because we know we have the ace of spades and then we just need to pick 4 other cards from the remaining 51 cards to fill out our hand. Then $P(A \cap B)$ is the probability of having a pair of aces with one of them being the ace of spades. To count this, we know that we have the ace of spades so there are $\binom{3}{1}$ ways to choose the other ace to complete the pair. Then out of the remaining 48 non-ace cards, we need to pick out 3 cards to fill out our hand. Therefore $P(A \cap B) = \frac{\binom{3}{1}\binom{48}{3}}{\binom{52}{5}}$. Therefore

$$P(A|B) = \frac{\binom{3}{1}\binom{48}{3}}{\binom{51}{4}}.$$

10. A red-green colorblind person picks an apple out of a bag. There are 4 red apples and 1 green apple. With probability 3/4 he says the correct color of the apple he picked out. What is the probability that he says that the apple he picks out is red?

Solution: The probability that he says red is the probability that the apple is red and he says red plus the probability that the apple is green and he says red. The first probability is $\frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}$. The second probability is $\frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$. So the probability that he says red is $\frac{3}{5} + \frac{1}{20} = \frac{13}{20}$.

11. In the previous apple problem, what is the probability that the apple is actually red when he says it is red?

> **Solution:** Let A be the event that the apple is red and let B be the event that he says red. Then we want to calculate $P(A|B) = \frac{P(A\cap B)}{P(B)}$. We calculated $P(B) = \frac{13}{20}$ from the previous problem. Then $P(A) = \frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}$. So $P(A|B) = \frac{12}{13}$. You can also solve this using Bayes Theorem.

2 Distributions

12. Suppose that I have a weighted die that lands on $1, 2, 3, 4, 5$ with equal probability and 6 5 times as likely as 1. Let X be the value of the die. What is the PMF for X?

> **Solution:** $f(1) = f(2) = f(3) = f(4) = f(5) = \frac{1}{10}, f(6) = \frac{1}{2}$.

13. For my weighted die in the previous problem, what is the probability in 10 rolls, I roll a 5 or 6 exactly 6 times? What kind of distribution is this?

> **Solution:** The probability of rolling a 5 or 6 is $\frac{1}{10} + \frac{1}{2} = \frac{3}{5}$. This is a binomial distribution and the probability is $\binom{10}{6}(3/5)^6(2/5)^4$.

14. For my weighted die in the previous problem, suppose that I keep rolling until I roll a 5 or 6. What is the expected number of times I need to roll the die? What kind of distribution is this?

> **Solution:** This is a geometric distribution. The expected number of times I need to roll the die, including the last roll, is $1 + \frac{1-p}{p} = 1 + \frac{2/5}{3/5} = \frac{5}{3}$.

15. Suppose that X is binomially distributed with $E[X] = 15$ and $Var(X) = 6$. How many trials n are there and what is the probability p of success?

> **Solution:** Since this is binomial, we know $np = 15$ and $np(1-p) = 6$ so $1-p = 2/5$ and $p = 3/5$ so $n = 15/(3/5) = 25$.

16. Suppose that the number of students who fill out course evaluations per day is Poisson distributed and on average 2 students fill out evaluations per day. What is the probability that in a week, no students fill out evaluations? What is the probability that in a week, 7 people fill out evaluations?

Solution: In a week, the number of students that fill out evaluations is Poisson distribution and on average there are $2 \cdot 7 = 14$ students that fill out evals. So $\lambda = 14$. The probability of having 0 students fill out evals is $f(0) = \frac{14^0 e^{-14}}{0!} = e^{-14}$. The probability of 7 students filling out evals is $f(7) = \frac{14^7 e^{-14}}{7!}$.

3 Hypothesis Testing/CLT

17. I have a (possibly biased) coin and flip it 100 times and get heads 90 times. What is the 95% confidence interval for p, the probability of flipping a heads?

Solution: The estimator for $\hat{p} = \frac{90}{100}$ and $\hat{\sigma} = \sqrt{\hat{p}(1-\hat{p})} = \frac{3}{10}$. Then the 95% confidence interval is $(\hat{p} - 2\hat{\sigma}/\sqrt{n}, \hat{p} + 2\hat{\sigma}/\sqrt{n}) = (0.9 - 2(0.3)/10, 0.9 + 2(0.3)/10) = (0.9 - 0.06, 0.9 + 0.06) = (0.84, 0.96)$.

18. When counting families with 2 children, I find that 83 of them have two girls, 102 of them have two boys, and 215 of them have one boy and one girl. Suppose that my null hypothesis is having a boy or girl is equally likely and the two children's genders are independent of each other. What kind of test should I use to test this hypothesis? Perform this test and explain what kind of table to look up as well as what value to look at.

Solution: This is a χ^2 test. The expected distribution is 100 families having 2 girls, 100 having two boys, and 200 having one boy and one girl. The χ^2 value is $\frac{(83-100)^2}{100} + \frac{(102-100)^2}{100} + \frac{(215-200)^2}{200} = 4.055$. We look up the critical value in a χ^2 table with $3 - 1 = 2$ degrees of freedom and compare it to 4.055. If the critical value is smaller, then we reject the null hypothesis.

19. Every day, the number of people who are born is Poisson distributed with an average of 4900 people per day. We count how many people are born in a span of 100 days and let $\bar{X}$ denote the average number of people born per day. What is the probability $P(\bar{X} \leq 4895)$?

Solution: Since it is Poisson distributed, $\lambda = 4900$ and so the standard deviation is $\sqrt{\lambda} = \sqrt{4900} = 70$. By the Central Limit Theorem, the average number of people born in a span of 100 days is approximately normally distributed with expected value $\lambda = 4900$ and standard deviation $70/\sqrt{n} = 70/\sqrt{100} = 7$. So, $P(\bar{X} \leq 4895) = 1/2 - P(4895 \leq \bar{X} \leq 4900) = 1/2 - z(\frac{|4895-4900|}{7}) = 1/2 - z(5/7)$, where we look up 5/7 in a z-score table.

20. What is the definition of variance? Prove that $Var(X) = E[X^2] - E[X]^2$.

Solution: By definition, $Var(X) = E[(X - E[X])^2]$.

$$E[(X - E[X])^2] = E[X^2 - 2XE[X] + E[X]^2] = E[X^2] - 2E[XE[X]] + E[E[X]^2].$$

Now $E[X]$ is a constant and $E[c] = c$ so we can get rid of expected values to get

$$= E[X^2] - 2E[X]E[X] + E[X]^2 = E[X^2] - E[X]^2.$$

21. What is the definition of covariance? Prove that $Cov(X,Y) = E[XY] - E[X]E[Y]$.

Solution:

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY - XE[Y] - YE[X] + E[X]E[Y]]$$

$$= E[XY] - E[XE[Y]] - E[YE[X]] + E[E[X]E[Y]].$$

Then since $E[X], E[Y]$ are constants and $E[cX] = cE[X]$, we can take them out to get

$$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y] = E[XY] - E[X]E[Y].$$

4 First Order Differential Equations

22. True **FALSE** A differential equation $y' = f(y,t)$ with an initial condition $y(0) = y_0$ will always have a unique solution.

23. Solve the differential equations $(t^3 + t^2)y' = \frac{t^2+2t+2}{2y}$ with the initial condition $y(1) = 1$.

Solution: This is first order but not linear so we use separable equations to get

$$2ydy = \frac{t^2+2t+2}{t^3+t^2}dt$$

We rewrite the right side using partial fractions ($t^3+t^2 = t^2(t+1)$) as $\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t+1}$. Solving for A, B, C gives $A = 0, B = 2, C = 1$ and integrating gives

$$y^2 = \int 2ydy = \int \frac{t^2+2t+2}{t^3+t^2}dt = \int \frac{2}{t^2}+\frac{1}{t+1}dt = \frac{-2}{t}+\ln|t+1|+C.$$

Now we plug in our initial condition $y(1) = 1$ to get $1 = -2+\ln 2+C$ and $C = 3-\ln 2$ so

$$y = \sqrt{-2/t+\ln|t+1|+3-\ln 2}.$$

24. Consider the differential equation $ty' + 3y = 5t^2$ with initial condition $y(1) = 1$. Draw a slope field and then estimate $y(5)$ using a step size of $h = 2$. Then solve for y explicitly and find the exact value of $y(5)$.

Solution: Moving things over, we get $y' = 5t-3y/t = f(t, y)$. Our initial condition is the point $(t_0, y_0) = (1, 1)$. Then our next point has $t_1 = 1 + h = 3$ and $y_1 = y_0+hf(t_0, y_0) = 1+2(5(1)-3(1/1)) = 5$. So we have the point $(3, 5)$. Now to get the next point we have $t_2 = t_1+h = 5$ and $y_2 = y_1+hf(t_1, y_1) = 5+2(5(3)-3(5/3)) = 25$. So the next point is $(5, 25)$ and $y(t)$ is about 25.

To find the exact solution, we need to solve for y. This is a first order linear equation so we use integrating factors. To do this, we write it as $y' + \frac{3}{t}y = 5t$. We multiply by the integrating factor which is $e^{\int 3/tdt} = e^{3\ln t} = t^3$ to get $t^3y'+3t^2y = 5t^4$. Integrating gives

$$t^3y = \int (t^3y)'dt = \int t^3y' + 3t^2ydt = \int 5t^4dt = t^5 + C.$$

Thus, we have that $y = t^2 + \frac{C}{t^3}$. Plugging in the initial condition $y(1) = 1$ gives $1 = 1 + C$ so $C = 0$ and $y = t^2$ is the solution so $y(5) = 25$.

25. Find all solutions to $e^t y' = y^2 + 2y + 1$.

Solution: This is a separable equation because we can write $y' = (y + 1)^2e^{-t}$. This gives $\frac{dy}{(Y+1)^2} = e^{-t}dt$ so integrating gives $\frac{-1}{y+1} = -e^{-t}+C$ so $\frac{1}{y+1} = e^{-t}+C$. Therefore, solving gives $y = \frac{1}{e^{-t}+C} - 1$. There is also a missing solution when we divided which was $y = -1$.

26. Newton's law of cooling says that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the ambient temperature. Suppose that I put a frozen pizza initially at $0°C$ into an oven with a temperature of $200°C$ and after half an hour the pizza has reached a temperature of $100°$. Find the temperature $T(t)$ of the pizza after t minutes.

Solution: Newton's law of cooling tells us that $T'(t) = k(T-200)$. Solving this via separable equations gives us that $\ln|T-200| = kt+C$ or $T(t) = Ce^{kt}+200$. Plugging in the initial condition gives us that $T(0) = 0 = C + 200$ so $T(t) = 200 - 200e^{kt}$. Then, we are told that $T(30) = 100$ so $200-200e^{30k} = 100$ or $e^{30k} = \frac{1}{2}$ and $k = \frac{\ln(1/2)}{30}$. Therefore $T(t) = 200 - 200e^{\ln(1/2)t/30} = 200 - 200(\frac{1}{2})^{t/30}$.

5 Recurrence Relations and 2nd order Differential Equations

27. **TRUE** False It is possible for an IVP to have a unique solution.
28. **TRUE** False It is possible for a BVP to have a unique solution.
29. True **FALSE** It is possible for an IVP to have infinitely many solutions.
30. **TRUE** False It is possible for a BVP to have infinitely many solutions.
31. Solve the recursion equation $a_n = 2a_{n-2} - a_{n-1}$ with the initial conditions $a_0 = 0, a_1 = 3$.

Solution: The characteristic equation is $\lambda^2 = 2-\lambda$ or $\lambda^2+\lambda-2 = (\lambda+2)(\lambda-1) = 0$ so $\lambda = 1, -2$ are the roots. Therefore, the general solution is $c_1(1)^n + c_2(-2)^n$ or $c_1 + c_2(-2)^n$. The initial conditions give $c_1 + c_2 = 0$ and $c_1 - 2c_2 = 3$. Adding twice the first to the second gives $3c_1 = 3$ so $c_1 = 1, c_2 = -1$. Therefore, the solution is $1 - (-2)^n$.

32. Verify that $y_1(t) = t$ and $y_2(t) = t^3$ are solutions to the differential equation $t^2y''(t) - 3ty'(t) + 3y(t) = 0$. Find the solution to the differential equation with $y(1) = 2$ and $y'(1) = 4$ (hint: what kind of differential equation is this?).

Solution: We plug in t and t^3 and show that we get an equality. Since this is a homogeneous linear polynomial, linear combinations of solutions are also solutions so $c_1t+c_2t^3$ is the general solution. Plugging in the initial condition gives $c_1+c_2 = 2$ and $c_1 + 3c_2 = 4$ so $c_1 = c_2 = 1$. Therefore, the solution is $y(t) = t + t^3$.

33. Find all solutions to the BVP $y'' + 2y' + 5y = 0$ with $y(0) = 0$ and $y(\pi) = 0$.

> **Solution:** The characteristic equation is $\lambda^2+2\lambda+5 = 0$ so the roots are $\lambda = -1\pm 2i$. Therefore, the general solution is $c_1e^{-t}\sin(2t) + c_2e^{-t}\cos(2t)$. Now plugging in the initial conditions give $c_2 = 0$ and $c_2e^{-\pi} = 0$ or $c_2 = 0$. Therefore, the solution is $y(t) = c_1e^{-t}\sin(2t)$ and there are infinitely many solutions.

34. Find all solutions to the BVP $y'' - 5y' + 6y = 0$ with $y(0) = 2$ and $y(1) = e^2 + e^3$.

> **Solution:** The characteristic equation is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$. So the roots are $\lambda = 2, 3$ and the general solution is $y(t) = c_1e^{2t} + c_2e^{3t}$. Plugging in the initial conditions give $c_1 + c_2 = 2, c_1e^2 + c_2e^3 = e^2 + e^3$ so $c_1 = c_2 = 1$. Therefore, the unique solution is $y(t) = e^{2t} + e^{3t}$.

35. Find a second order differential equation IVP that has te^t as a solution.

> **Solution:** Since te^t is a solution, the t in front tells us that there is a double root and the e^t tells us that $\lambda = 1$ is a root. Therefore, the roots are $\lambda = 1, 1$. So the characteristic equation is $(\lambda - 1)(\lambda - 1) = \lambda^2 - 2\lambda + 1 = 0$. So the differential equations is $y'' - 2y' + y = 0$. The conditions are $y(0) = 0e^0 = 0, y'(0) = te^t + e^t|_{t=0} = 0e^0 + e^0 = 1$.

36. Find a second order differential equation BVP that has $e^{2t}\sin(t)$ as a solution.

> **Solution:** Since we have sin, we know that there are complex roots so the roots are $\lambda = a \pm bi$. The a is the exponent of e^{2t} so $a = 2$ and b is in the sin or cos so $b = 1$. Therefore, $\lambda = 2 \pm i$ are the roots. So the characteristic equation is $(\lambda - (2 - i))(\lambda - (2 + i)) = 0$ or $\lambda^2 - 4\lambda + 5 = 0$. Therefore, the differential equation is $y'' - 4y' + 5y = 0$.
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> The initial conditions is a boundary value so we could take $y(0) = e^0\sin(0) = 0$ and $y(1) = e^2\sin(1)$.

6 Matrices and Correlation

37. True **FALSE** If A, B are square $n \times n$ matrices, then $AB = BA$.

38. True **FALSE** If A is a 2×2 matrix such that $A^2 = I_2$, then $A = I_2$.

39. Find the solution to

$$\begin{cases} y_1'(t) = -y_1(t) - 5y_2(t) \\ y_2'(t) = 2y_1(t) + y_2(t) \end{cases}$$

with $y_1(0) = -5$ and $y_2(0) = -2$.

Solution: We can write this as $\vec{y}' = A\vec{y}$ where $A = \begin{pmatrix} -1 & -5 \\ 2 & 1 \end{pmatrix}$. The characteristic equation is $\lambda^2 + 9 = 0$ so $\lambda = \pm 3i$ are the roots. For $\lambda = 3i$, we have $A - \lambda I = \begin{pmatrix} -1-3i & -5 \\ 2 & 1-3i \end{pmatrix}$ so a eigenvector is $\begin{pmatrix} -5 \\ 1+3i \end{pmatrix}$. Therefore, a solution is

$$e^{3it}\begin{pmatrix} -5 \\ 1+3i \end{pmatrix} = (\cos 3t + i \sin 3t)\begin{pmatrix} -5 \\ 1+3i \end{pmatrix}$$

$$= \begin{pmatrix} -5\cos 3t - 5i \sin 3t \\ \cos 3t - 3\sin 3t + i \sin 3t + 3i \cos 3t \end{pmatrix} = \begin{pmatrix} -5\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} + i \begin{pmatrix} -5\sin 3t \\ \sin 3t + 3\cos 3t \end{pmatrix}$$

Therefore the general solution is a linear combination of the real and complex parts and it is

$$c_1 \begin{pmatrix} -5\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} -5\sin 3t \\ \sin 3t + 3\cos 3t \end{pmatrix}$$

Plugging in $t = 0$ gives us that

$$c_1 \begin{pmatrix} -5 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}.$$

Solving gives $c_1 = 1, c_2 = -1$ so the solution is

$$y(t) = \begin{pmatrix} -5\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} - \begin{pmatrix} -5\sin 3t \\ \sin 3t + 3\cos 3t \end{pmatrix} = \begin{pmatrix} 5\sin 3t - 5\cos 3t \\ -2\cos 3t - 4\sin 3t \end{pmatrix}$$

40. Consider the following set of points: $\{(0,6),(1,3),(2,1),(3,0),(4,0)\}$. Find the line of best fit through these points and use it to estimate $y(0.5)$. What is the correlation of the data?

Solution: We can calculate the line $y = ax+b$ as $a = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{-15}{10} = \frac{-3}{2}$ and $b = \bar{y} - a\bar{x} = 2 - 2(-3/2) = 5$. So the lineof best fit is $y = 5 - 3/2x$. We have $y(0.5) \approx 5 - 3/2(1/2) = 5 - 3/4 = 4.25$.

The correlation is $r = \frac{\sum(x_i - \bar{x})(y - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2}\sqrt{\sum(y_i - \bar{y})}} = \frac{-15}{\sqrt{10}\sqrt{26}} = \frac{-15}{\sqrt{260}}$.

41. Let $A = \begin{pmatrix} 3 & 4 & -1 \\ 4 & 2 & 1 \\ -2 & -3 & 1 \end{pmatrix}$. Find A^{-1}.

Solution: Use Gaussian elimination to get $A^{-1} = \begin{pmatrix} -5 & 1 & -6 \\ 6 & -1 & 7 \\ 8 & -1 & 10 \end{pmatrix}$.

42. Let A be the same as the previous problem. Solve $A\vec{x} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ (hint: use the previous problem to do this quickly).

Solution: The solution is $\vec{x} = A^{-1}\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ and we calculated A^{-1} in the previous problem to get $\vec{x} = \begin{pmatrix} 5 \\ -5 \\ -6 \end{pmatrix}$.

43. Let $\vec{v}_1 = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ and suppose that A is a 3×3 matrix such that $A\vec{v}_1 = 4\vec{v}_1, A\vec{v}_2 = \vec{0}, A\vec{v}_3 = -\vec{v}_3$. What are the eigenvalues and eigenvectors of A? What is the general solution to $\vec{y}'(t) = A\vec{y}$ with $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$?

Solution: The eigenvalues are $4, 0, -1$ with eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ respectively. The general solution is $e^{4t}\vec{v}_1 + e^{0t}\vec{v}_2 + e^{-t}\vec{v}_3$.

44. Let A be a 2×2 matrix and suppose that $\vec{y} = \begin{pmatrix} 3e^{2t} + 4e^{4t} \\ e^{4t} - e^{2t} \end{pmatrix}$ is a solution to $\vec{y}' = A\vec{y}$. What are the eigenvalues and eigenvectors of A? What is $A\begin{pmatrix} 3 \\ -1 \end{pmatrix}, A\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 7 \\ 0 \end{pmatrix}$?

Solution: We write the solution as $e^{2t}\begin{pmatrix}3\\-1\end{pmatrix}+e^{4t}\begin{pmatrix}4\\1\end{pmatrix}$. Therefore, one eigenvalue is 2 with eigenvector $\begin{pmatrix}3\\-1\end{pmatrix}$ and the other is 4 with eigenvector $\begin{pmatrix}4\\1\end{pmatrix}$. Then $A\begin{pmatrix}3\\-1\end{pmatrix} = 2\begin{pmatrix}3\\-1\end{pmatrix}$ and $A\begin{pmatrix}4\\1\end{pmatrix} = 4\begin{pmatrix}4\\1\end{pmatrix}$ and $A\begin{pmatrix}7\\0\end{pmatrix}$ is the sum.

45. Find the line of best fit through the points $\{(0,2),(1,3),(2,1)\}$ and the correlation of the data.

Solution: We calculate $\bar{x}=1, \bar{y}=2$ and the line $y=ax+b$ is $a = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2} = \frac{-1}{2}$ and $b=\bar{y}-a\bar{x} = 2-1(-1/2) = 2.5$. So the line of best fit is $y = 2.5 - x/2$. The correlation is $r = \frac{\sum(x_i-\bar{x})(y-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2}\sqrt{\sum(y_i-\bar{y})}} = \frac{-1}{\sqrt{2}\sqrt{2}} = \frac{-1}{2}$.

46. Write the differential equation $y''+5y'+6y=0$ as a systems of differential equations with $y_1(t)=y(t), y_2(t)=y'(t)$ and solve with $y(0)=2, y'(0)=-5$.

Solution: We can write it as $y_1'(t) = y_2(t)$ and $y_2'(t) = -6y_1(t)-5y_2(t)$ so it is represented in the matrix $\begin{pmatrix}0 & 1\\-6 & -5\end{pmatrix}$. Solving gives us that $y_1(t)=y(t)=e^{-2t}+e^{-3t}$.

47. Suppose that $a_{n+1} = 3a_n - 4a_{n-1}$. Let $\vec{v}_n = \begin{pmatrix}a_n\\a_{n+1}\end{pmatrix}$. Find a matrix A such that $\vec{v}_{n+1} = A\vec{v}_n$.

Solution: By looking at the dimensions A has to be a 2×2 matrix. Then we want

$$\begin{pmatrix}a_{n+1}\\a_{n+2}\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}a_n\\a_{n+1}\end{pmatrix} = \begin{pmatrix}aa_n+ba_{n+1}\\ca_n+da_{n+1}\end{pmatrix}.$$

By comparing coefficients, we get that $a=0, b=1$, and then $c=-4, d=3$. So $A = \begin{pmatrix}0 & 1\\-4 & 3\end{pmatrix}$.